# Control and Design for Fluids and Structures

Enrique Zuazua

Ikerbasque & Basque Center for Applied Mathematics (BCAM)
Bilbao, Basque Country, Spain
zuazua@bcamath.org
http://www.bcamath.org/zuazua/

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(bcam)
basque center for applied mathematics


Research Lines
PDE
partial differential equations, numerics and control
NET
network design, analysis and optimization
CM
computational mathematics
CFD
Computational fluid dynamics
RLI
research line incubator
MBMS
mathematical biology & molecular simulation

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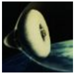
# Control theory and applications

**Application domains of control theory:**

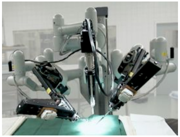
## Mechanics

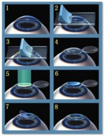

Vehicles (guidance, dampers, ABS, ESP, ...),
Aeronautics, aerospace (shuttle, satellites), robotics



## Biology, medicine

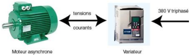
Predator-prey systems, bioreactors, epidemiology,
medicine (peacemakers, laser surgery)





## Electricity, electronics

RLC circuits, thermostats, regulation, refrigeration, computers, internet
and telecommunications in general, photography and digital video



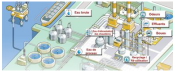


## Economics

Gain optimization, control of financial flux,
Market prevision

## Chemistry

Chemical kinetics, engineering process, petroleum, distillation, petrochemical industry



## Control theory

The main issues are:

**Controllability**

Steer the system from an initial state to a prescribed final one.

**Optimal control**

One aims moreover at minimizing some criterion, minimal cost of control for instance.

**Stabilization**

Once a trajectory has been planned, one aims at stabilizing it in order to make it robust, insensitive to perturbations, by means of feedback controls.

**Observability and parameter identification**

Recover the complete state from partial information.

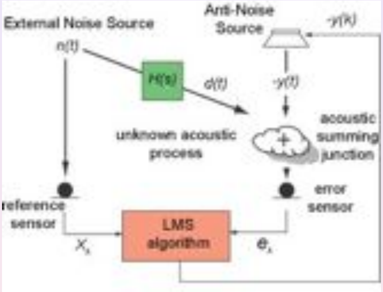
# THE ORIGINS:

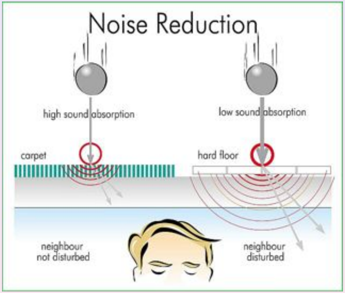
> *"... if every instrument could accomplish its own work, obeying or anticipating the will of others ...*
> *if the shuttle weaved and the pick touched the lyre without a hand to guide them, chief workmen would not need servants, nor masters slaves."*

Chapter 3, Book 1, of the monograph "Politics" by Aristotle (384-322 B. C.).

Main motivation: The need of automatizing processes to let the human being gain in liberty, freedom, and quality of life.

# An example: noise reduction



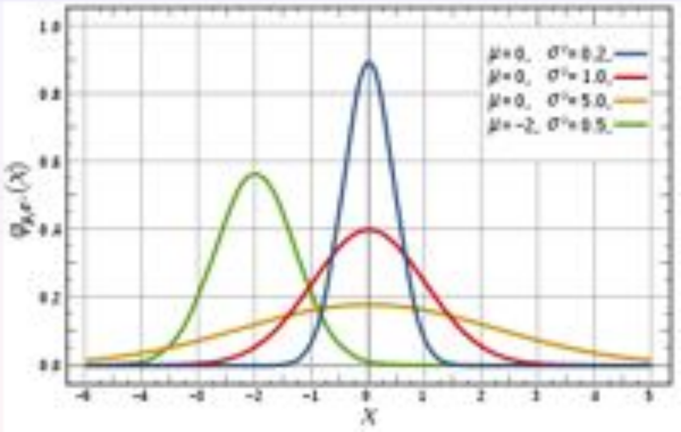

Acoustic noise reduction

## An example: noise reduction

Noise reduction is a subject to research in many different fields. Depending on the environment, the application, the source signals, the noise, and so on, the solutions look very different. Here we consider noise reduction for audio signals, especially speech signals, and concentrate on common acoustic environments such an office room or inside a car. The goal of the noise reduction is to reduce the noise level without distorting the speech, thus reduce the stress on the listener and - ideally - increase intelligibility.

# Other applications of noise reduction

## Gaussian filters:




$$u(x) = [G(\cdot) \star f(\cdot)](x); \quad G(x) = (4\pi)^{-N/2} \exp(-|x|^2/4).$$

*Control in an information rich World, SIAM, R. Murray Ed., 2003.*

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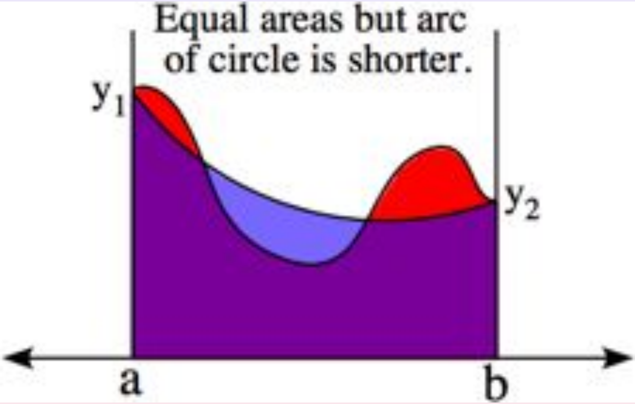
## The Calculus of Variations



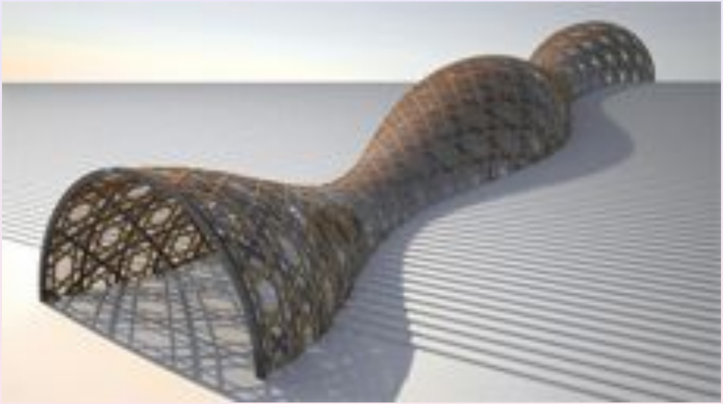

x

**Calculus of variations** deals with maximizing or minimizing functionals, as opposed to ordinary calculus which deals with maximizing and minimizing ordinary functions.

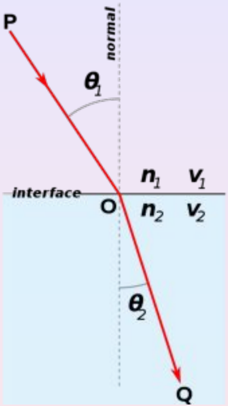
- The curve of shortest length, **geodesic**, connecting two points.
- **Fermat's principle:** light follows the path of shortest optical length connecting two points, where the optical length depends upon the material of the medium.

  *Leonhard Euler(1707-1783): For since the fabric of the universe is most perfect and the work of a most wise creator, nothing at all takes place in the universe in which some rule of the maximum or minimum does not appear.*

## Geodesic curves

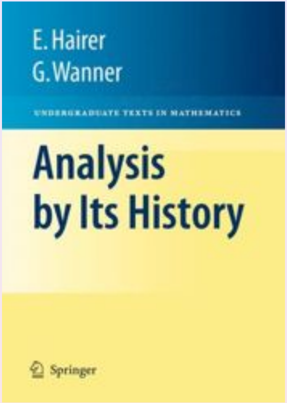


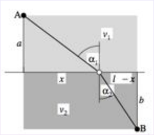
## Fermat's principle/Snellious law




$$\frac{\sin(\theta_1)}{\sin(\theta_2)} = \frac{v_1}{v_2} = \frac{n_2}{n_1}.$$

Named after the dutch astronomer Willebrord Snellius (1580 - 1626).
Pierre de Fermat (1601 - 1665).


E. Hairer
G. Wanner
UNDERGRADUATE TEXTS IN MATHEMATICS
Analysis
by Its History
Springer




To find $x$ s. t.

$$T = \frac{\sqrt{a^2 + x^2}}{v_1} + \frac{\sqrt{b^2 + (\ell - x)^2}}{v_2}.$$

Fermat found the problem too difficult for an analytical treatment (*I admit that this problem is not one of the easiest*). The computations were then proudly performed by Leibniz (1684)

$$T' = \frac{1}{v_1}\frac{2x}{2\sqrt{a^2+x^2}} - \frac{1}{v_2}\frac{2(\ell - x)}{2\sqrt{b^2+(\ell-x)^2}}.$$

Observing that $\sin(\alpha_1) = x/\sqrt{a^2+x^2}$; $\sin(\alpha_2) = (\ell - x)/\sqrt{b^2+(\ell-x)^2}$ we see that this derivative vanishes whenever

$$\frac{\sin(\theta_1)}{\sin(\theta_2)} = \frac{v_1}{v_2}.$$

Furthermore:

$$T'' = \frac{1}{v_1}\frac{a^2}{(a^2+x^2)^{3/2}} + \frac{1}{v_2}\frac{b^2}{(b^2+(\ell-x)^2)^{3/2}} > 0,$$

showing that the critical point is the minimizer.

## Optimal transport

In mathematics and economics, transportation theory refers to the study of optimal transportation and allocation of resources. The problem was formalized by the French mathematician Gaspard Monge in 1781 ("Sur la théorie des déblais et des remblais" (Mém. de l'Acad. de Paris, 1781))

**An example:** Suppose that we have $n$ books of equal width on a shelf (the real line), arranged in a single contiguous block. We wish to rearrange them into another contiguous block, but shifted one book-width to the right. Two obvious candidates for the optimal transport plan present themselves:

- Move all $n$ books one book-width to the right; ("many small moves")
- Move the left-most book $n$ book-widths to the right and leave all other books fixed. ("one big move")

If the cost function is proportional to Euclidean distance $(c(x,y) = \alpha|x-y|)$ then these two candidates are both optimal. If, on the other hand, we choose the strictly convex cost function proportional to the square of Euclidean distance $(c(x,y) = \alpha|x-y|^2)$, then the "many small moves" option becomes the unique minimizer.

But the origins of the potential applications of the idea of optimal transport and geodesic paths goes back to the ancient Egipt where the "harpenodaptai" had as main task drawing long straight lines on the sand of the desert.

## Minimal surfaces

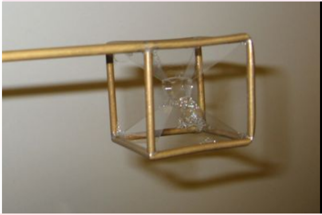
Minimal surfaces are defined as surfaces with zero mean curvature.

Finding a minimal surface of a boundary with specified constraints is a problem in the Calculus of Variations and is sometimes known as **Plateau's problem**.

Physical models of area-minimizing minimal surfaces can be made by dipping a wire frame into a soap solution, forming a soap film, which is a minimal surface whose boundary is the wire frame.

Enneper's surface:

$$x = u(1 - u^2/3 + v^2)/3; y = -v(1 - v^2/3 + u^2)/3; z = (u^2 - v^2)/3.$$


## Isoperimetric inequalities

Isoperimetric literally means "having the same perimeter". The isoperimetric problem is to determine a plane figure of the largest possible area whose boundary has a specified length. The isoperimetric inequality states, for the length $L$ of a closed curve and the area $A$ of the planar region that it encloses, that

$$4\pi A \leq L^2,$$

and that equality holds if and only if the curve is a circle.

The closely related Dido's problem asks for a region of the maximal area bounded by a straight line and a curvilinear arc whose endpoints belong to that line. It is named after Dido, the legendary founder and first queen of Carthage.

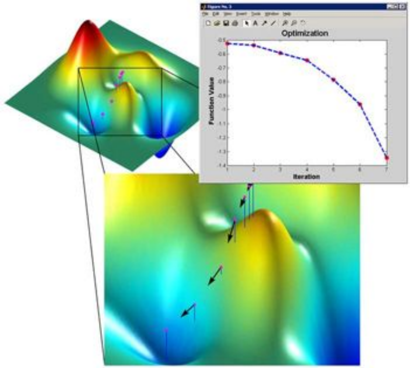
Carthage & Cologne

# The computational version of the Calculus of Variations

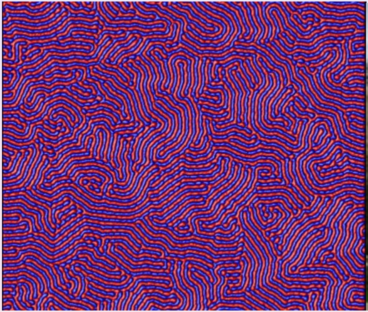
$$J(u^*) = \min_{u \in \mathcal{U}} J(u).$$

**Gradient methods:**

$$u_{k+1} = u_k - \rho \nabla J(u_k).$$



## Montecarlo methods:



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"Cybernétique" was proposed by the French physicist A.-M. Ampère in the XIX Century to design the nonexistent science of process controlling. This was quickly forgotten until 1948, when Norbert Wiener (1894–1964) chose "Cybernetics" as the title of his famous book.

Wiener defined Cybernetics as " the science of control and communication in animals and machines".

In this way, he established the connection between Control Theory and Physiology and anticipated that, in a desirable future, engines would obey and imitate human beings.

Let $n, m \in \mathbb{N}^*$ and $T > 0$ and consider the following linear finite-dimensional system

$$x'(t) = Ax(t) + Bu(t), \quad t \in (0, T); \quad x(0) = x^0. \tag{1}$$

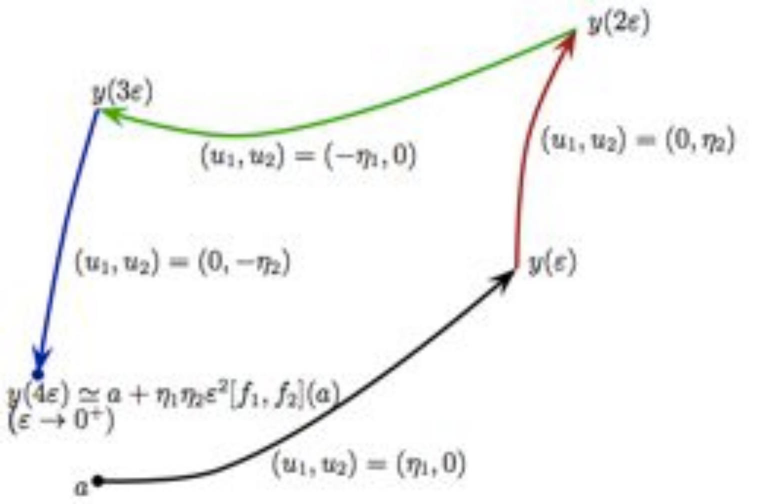
In (1), $A$ is a $n \times n$ real matrix, $B$ is of dimensions $n \times m$ and $x^0$ is the initial sate of the sytem in $\mathbb{R}^n$. The function $x : [0, T] \longrightarrow \mathbb{R}^n$ represents the*state* and $u : [0, T] \longrightarrow \mathbb{R}^m$ the *control*.

¿Can we control the state $x$ of $n$ components with only $m$ controls, even if $n >> m$?

**Theorem**

*(1958, Rudolf Emil Kálmán (1930– )) System* (1) *is controllable iff*

$$rank\,[B,\, AB, \cdots, A^{n-1}B] = n.$$




J. M. Coron, BCAM, June 2011.

## Proof:

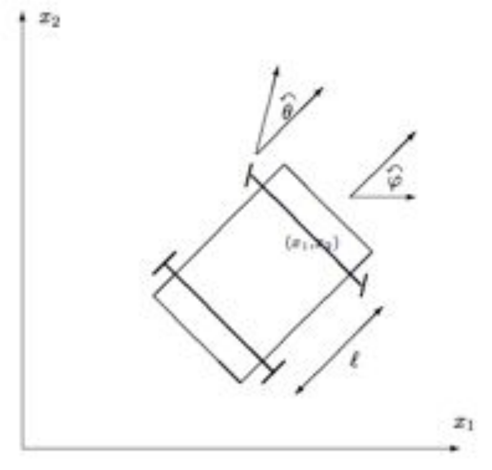
From the variation of constants formula:

$$x(t) = e^{At}x^0 + \int_0^t e^{A(t-s)}Bu(s)ds = e^{At}x^0 + \int_0^t \sum_{k\geq 0} \frac{(t-s)^k}{k!}A^k Bu(s)ds.$$

By Cayley[1]-Hamilton's [2] theorem $A^k$ for $k \geq n$ is a linear combination of $I, A, ..., A^{n-1}$.

[1] Arthur Cayley (UK, 1821 - 1895)
[2] William Rowan Hamilton (Ireland, 1805 - 1865)

## An example: Nelson's car.




Two controls suffice to control a four-dimensional dynamical system.

E. Sontag, *Mathematical control theory*, 2nd ed.,Texts in Applied Mathematics,vol.6, Springer-Verlag, NewYork,1998.

## An example: Parking a car.

In practice the difficulty: wheel constraints

## An example: Parking a car.

Increasing complexity

# An example: Parking a car.

Optimality: minimizing time

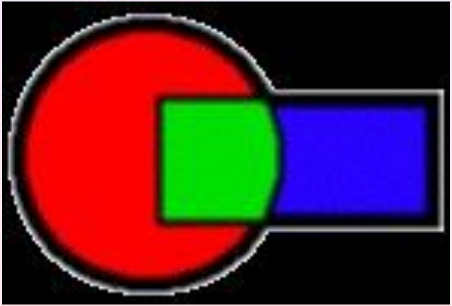
## An example: Parking a car.

The perfect parallel parking

The norwegian mathematician Marius Sophus Lie ( 1842 – 1899) observed that:

$$\exp(A+B) = \lim_{n\to\infty} \Big[ \exp(A/n) \exp(B/n) \Big]^n.$$

The same idea inspired Karl Hermann Amandus Schwarz (1843 – 1921) when introducing the nowadays ubiquitous method of *Domain Decomposition*:



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# The optimal pancake

Mail Online

## The perfect pancake? Easy, just follow this formula ... 100 - [10L - 7F + C(k - C) + T(m - T)]/(S - E)

By Daily Mail Reporter
Last updated at 9:49 AM on 24th February 2009

With Shrove Tuesday tomorrow it was perhaps inevitable that an eager scientist would apply their skills to creating the perfect pancake.

Maths expert Dr Ruth Fairclough stepped up to the challenge, unveiling a complex algebra formula to help chefs nail the dish on the day.

The 34-year-old senior lecturer of mathematics and statistics worked out the food formula because her two daughters loved eating pancakes so much.

Dr Ruth, who teaches at Wolverhampton University found that 100 - [10L - 7F + C(k - C) + T(m - T)]/(S - E) created the tastiest snack.

In the complex formula L represents the number of lumps in the batter and C equals its consistency.

The letter F stands for the flipping score, k is the ideal consistency and T is the temperature of the pan.

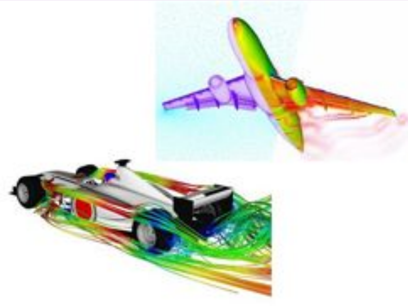
Ideal temp of pan is represented by m, S is the length of time the batter stands before cooking and E is the length of time the cooked pancake sits before being eaten.

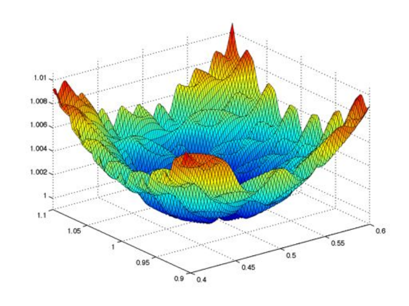
## Optimal shape design in aeronautics.

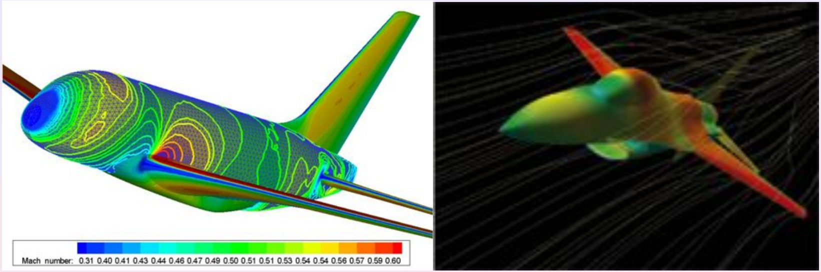
- Objective: To modify the shape of the airplane so to improve its efficiency, security, reduce noise, energy consumption, reduce drag, augment lift,...
- Point of view: That of the wind tunnel. The airplane is fixed while air is flowing around.
- Variations: When modifying the shape of the airplane, the way air is flowing around is modified, and the pressure field it applies into the airplane as well. The aerodynamical properties of the airplane are modified.


## Tools.

- Computational fluid mechanics: It allows to simulate the flow of air around a cavity.
- Optimization: It allows building an iterative algorithm to improve performance.








Computed pressure field over the surface of the airplane and flow lines of particles of air.

The method consists on formulating the problem in the context of the Calculus of Variations. To minimize

$$J(\Omega^*) = \min_{\Omega \in \mathcal{C}_{ad}} J(\Omega)$$

where $\mathcal{C}_{ad}$ is the class of admissible shapes $\Omega$, and $J =$ is the cost functional measuring the efficiency of the design (drag, lift,...)

$J$ depends on $\Omega$ but not directly, rather thorugh $u(\Omega)$, the solution of the air-dynamics in the exterior of the airplane.

## Leonhard Euler

(1707-1783) derived the equations for the motion of perfect fluids, in the absence of viscosity:

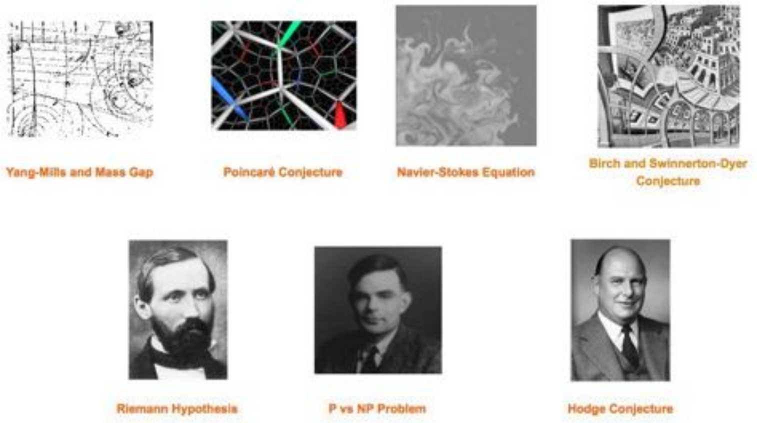
$$u_t + u \cdot \nabla u = \nabla p.$$

But D'Alembert observed that the flight of birds would be impossible according to that model.
Claude Louis Marie Henri Navier (1785-1836) and Sir George Gabriel Stokes (1819-1903) much later incorporated the viscosity term:

$$u_t - \nu \Delta u + u \cdot \nabla u = \nabla p.$$

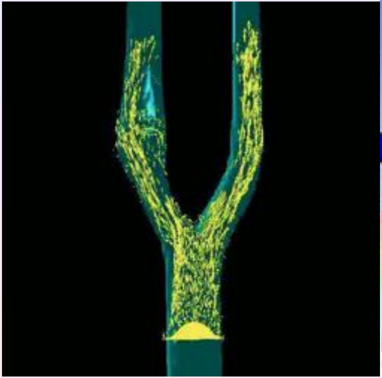
There are many open complex problems in the field of Fluid Mechanics.

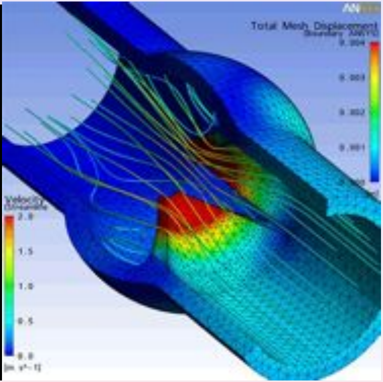
Fluid Mechanics is one of the most important areas of Physics because of its impact on our life : air, water, blood,...



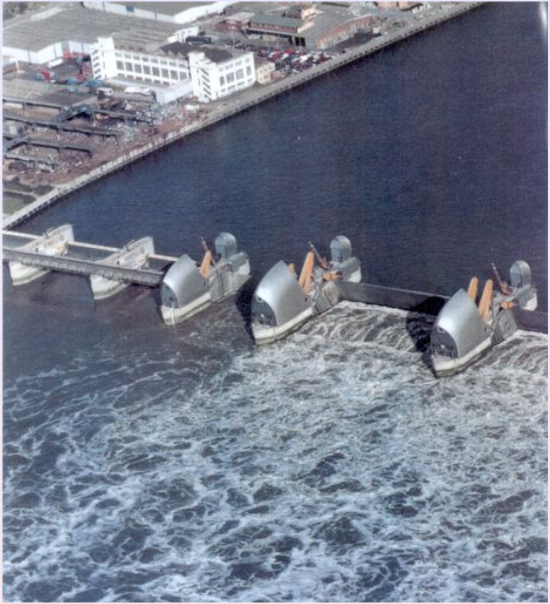

The millenium problems

The science program is still ongoing to a large extent thanks to computers.





Pascalina, Blaise Pascal, 1645; ENIAC: Electronic Numerical Integrator And Computer, 1946; Macbook Air, 2008.



The Thames Barrier

# An example: Swimming fishes.

Swimming fishes

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In mathematics, computational science, or management science, mathematical optimization (alternatively, optimization or mathematical programming) refers to the selection of a best element from some set of available alternatives.

- Convex programming
- Linear programming
- Semidefinite programming
- Conic programming
- Stochastic programming
- Robust programming
- Combinatorial optimization
- Dynamic programming
- Heuristics and metaheuristics
- ....

An example in logistics This is a typical and ubiquitous example in linear programming. A company $s_i$, $i \leq 1 \leq M$ items in each of the $M$ storage locations. $N$ clients request $r_j$ items each, $1 \leq j \leq N$. The cost of transportation between the $i$-th storage location and the $j$-th client is $c_{ij}$. We have to decide about the number of items to be delivered from the $i$-th storage location the $j$-th client, $v_{ij}$.

Of course we want to minimize the cost of transportation. The problem is then that of minimizing the functional

$$\inf_{\{v_{ij}\}} \left( \sum_{i=1}^{M} \sum_{j=1}^{N} c_{ij} v_{ij} \right)$$

under the constraints

$$v_{ij} \geq 0; \sum_{j=1}^{N} v_{ij} \leq s_i; \sum_{i=1}^{M} v_{ij} = r_j, 1 \leq i \leq M; 1 \leq j \leq N.$$

# These tools are so much used that nowadays there is plenty of software available both free and comercial: IPOPT

## Welcome to the Ipopt home page

*Note that these project webpages are based on Wiki, which allows webusers to modify the content to correct typos, add information, or share their experience and tips with other users. You are welcome to contribute to these project webpages. To edit these pages or submit a ticket you must first register and login.*

### Introduction

Ipopt (**I**nterior **P**oint **OPT**imizer, pronounced eye-pea-Opt) is a software package for large-scale nonlinear optimization. It is designed to find (local) solutions of mathematical optimization problems of the from

```
   min     f(x)
x in R^n

s.t.       g_L <= g(x) <= g_U
           x_L <=  x   <= x_U
```

where `f(x): R^n --> R` is the objective function, and `g(x): R^n --> R^m` are the constraint functions. The vectors `g_L` and `g_U` denote the lower and upper bounds on the constraints, and the vectors `x_L` and `x_U` are the bounds on the variables `x`. The functions `f(x)` and `g(x)` can be nonlinear and nonconvex, but should be twice continuously differentiable. Note that equality constraints can be formulated in the above formulation by setting the corresponding components of `g_L` and `g_U` to the same value.

### Background

Ipopt is written in C++ and is released as open source code under the Eclipse Public License (EPL). It is available from the COIN-OR initiative. The code has been written by Carl Laird and Andreas Wächter, who is the COIN project leader for Ipopt.

The Ipopt distribution can be used to generate a library that can be linked to one's own C++, C, or Fortran code, as well as a solver executable for the AMPL modeling environment. The package includes interfaces to CUTEr optimization testing environment, as well as the MATLAB and R programming environments. IPOPT can be used on Linux/UNIX, Mac OS X and Windows platforms.

As open source software, the source code for Ipopt is provided without charge. You are free to use it, also for commercial purposes. You are also free to modify the source code (with the restriction that you need to make your changes public if you decide to distribute your version in any way, e.g. as an executable); for details see the CPL license. And we are certainly very keen on feedback from users, including contributions!

In order to compile Ipopt, certain third party code is required (such as some linear algebra routines, or the AMPL Solver Library). Those are available under different conditions/licenses.

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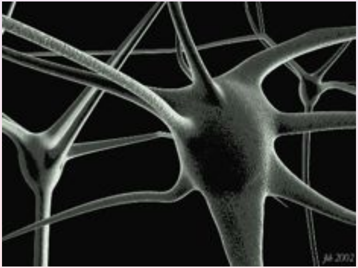
Mathematics are and will be increasingly influenced by the challenge of dealing with complexity and multidisciplinarity. The following areas will gain relevance:

- Discrete mathematics, combinatorics, graphs,...;
- Data mining;
- Statistical learning;

and other fields of research such as neurosciences and social sciences.



# A mathematician is a machine for turning coffee into theorems

**The Erdös Number Project**

**This is the website for the Erdös Number Project, which studies research collaboration among mathematicians.**

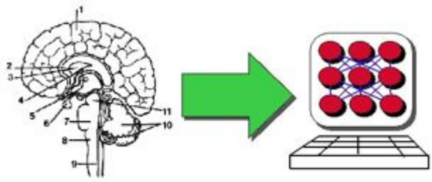
*The site is maintained by* ***Jerry Grossman*** *at* ***Oakland University****.* ***Patrick Ion****, a retired editor at* ***Mathematical Reviews****, and* ***Rodrigo De Castro*** *at the* ***Universidad Nacional de Colombia, Bogota*** *provided assistance in the past. Please address all comments, additions, and corrections to Jerry at* ***grossman@oakland.edu****.*

**Erdös numbers** have been a part of the **folklore of mathematicians** throughout the world for many years. For an introduction to our project, a description of what Erdös numbers are, what they can be used for, who cares, and so on, choose the "What's It All About?" link below. To find out who **Paul Erdös** is, look at this **biography** at the MacTutor History of Mathematics Archive, or choose the "Information about Paul Erdös" link below. Some useful information can also be found in **this Wikipedia article**, which may or may not be totally accurate.

Paul Erdös (1913–1996) was a Hungarian mathematician. He published more papers than any other mathematician in history, working with hundreds of collaborators. He worked on problems in combinatorics, graph theory, number theory, classical analysis, approximation theory, set theory, and probability theory.

# There will be unexpected advances in computing algorithms...

| Year | Development | Key early figures |
|---|---|---|
| 263 | Gaussian elimination | Liu, Lagrange, Gauss, Jacobi |
| 1671 | Newton's method | Newton, Raphson, Simpson |
| 1795 | Least-squares fitting | Gauss, Legendre |
| 1814 | Gauss quadrature | Gauss, Jacobi, Christoffel, Stieltjes |
| 1855 | Adams ODE formulas | Euler, Adams, Bashforth |
| 1895 | Runge–Kutta ODE formulas | Runge, Heun, Kutta |
| 1910 | Finite differences for PDE | Richardson, Southwell, Courant, von Neumann, Lax |
| 1936 | Floating-point arithmetic | Torres y Quevedo, Zuse, Turing |
| 1943 | Finite elements for PDE | Courant, Feng, Argyris, Clough |
| 1946 | Splines | Schoenberg, de Casteljau, Bezier, de Boor |
| 1947 | Monte Carlo simulation | Ulam, von Neumann, Metropolis |
| 1947 | Simplex algorithm | Kantorovich, Dantzig |
| 1952 | Lanczos and CG iterations | Lanczos, Hestenes, Stiefel |
| 1952 | Stiff ODE solvers | Curtiss, Hirschfelder, Dahlquist, Gear |
| 1954 | Fortran | Backus |
| 1958 | Orthogonal linear algebra | Aitken, Givens, Householder, Wilkinson, Golub |
| 1959 | Quasi-Newton iterations | Davidon, Fletcher, Powell, Broyden |
| 1961 | QR algorithm for eigenvalues | Rutishauser, Kublanovskaya, Francis, Wilkinson |
| 1965 | Fast Fourier transform | Gauss, Cooley, Tukey, Sande |
| 1971 | Spectral methods for PDE | Chebyshev, Lanczos, Clenshaw, Orszag, Gottlieb |
| 1971 | Radial basis functions | Hardy, Askey, Duchon, Micchelli |
| 1973 | Multigrid iterations | Fedorenko, Bakhvalov, Brandt, Hackbusch |
| 1976 | EISPACK, LINPACK, LAPACK | Moler, Stewart, Smith, Dongarra, Demmel, Bai |
| 1976 | Nonsymmetric Krylov iterations | Vinsome, Saad, van der Vorst, Sorensen |
| 1977 | Preconditioned matrix iterations | van der Vorst, Meijerink |
| 1977 | MATLAB | Moler |
| 1977 | IEEE arithmetic | Kahan |
| 1982 | Wavelets | Morlet, Grossmann, Meyer, Daubechies |
| 1984 | interior-point methods | Fiacco, McCormick, Karmarkar, Megiddo |
| 1987 | Fast multipole method | Rokhlin, Greengard |
| 1991 | Automatic differentiation | Iri, Bischof, Carle, Griewank |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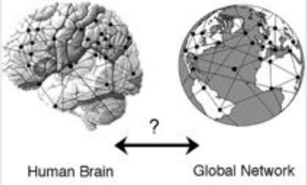
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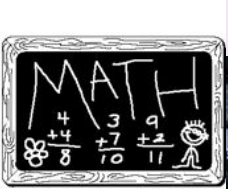
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