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Fuzzy tuning systems: the mathematics of musicians☆

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Abstract

We present some mathematical properties which determine tuning methods. We introduce the concept of fuzzy tuning systems and we analyze four of the systems coexisting within the current orchestras: Pythagorean, Just Intonation, Hölder's and Equal Temperament systems. We show that the theoretical and practical tuning methods are the same. We introduce the idea of compatibility between tuning systems and we give some sufficient conditions to determine an appropriate number of notes into which the octave must be divided.

Keywords: Tuning systems; Fuzzy sets; Fuzzy numbers; Continued fractions

1. Introduction

A tuning system is the set of sounds that music uses. By this, we mean that from the set of the frequencies of all the possible sounds, $\mathbb{R}^+$, a subset containing the appropriate frequencies is selected. Different criteria have been used to make this selection but, at least since the 4th century B.C., most tuning systems have been obtained by means of mathematical arguments [10–12]. It is undeniable that the numerical nature of these systems made instrument manufacturing easier and also facilitated their transmission [7]. However, the crispness of the mathematical arguments relegated these tuning systems to theoretical studies, while in practice musicians tuned in a more flexible way.

In fact, if we represent graphically the frequencies at an instant t produced by each one of the instruments in an orchestra, the great differences observed between sounds considered as well tuned would be surprising. Nevertheless, the *ensemble sensation* is very pleasant [15]. This phenomenon

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allows us to introduce the concept of compatibility between notes by means of an index of consistency, introduced by Zadeh [17], which measures how possible it is for the fuzzy numbers associated to the notes to be equal.

In this paper, we show that, as in many other human activities, what musicians do is to apply fuzzy decision rules for their selection criteria [8,18]. Actually, we will see that the theoretical tuning systems and the well-tuned sounds (as tested by a chromatic tuner) that musicians use in practice are the same.

On the other hand, the compatibility between two notes becomes insufficient when the tuning of more than one instrument is analyzed. In this case, it is necessary to study when two tuning systems can coexist and the concept of α-compatibility between tuning systems appears naturally, where α represents the level of similarity between these systems.

2. Previous concepts

In this paper, we will identify each musical note with the frequency of its fundamental harmonic (the frequency that tuners measure) because we will work with tuning systems. The usual way to relate two frequencies is through their ratio and this number is called the *interval*. It is well known that, in the middle zone of the audible field, the "pitch sensation" changes approximately according to the logarithm of the frequency, so the distance between two notes sounds whose frequencies are f_1 and f_2 can be estimated by means of the expression

$$d(f_1, f_2) := 1200 \left| \log_2 \frac{f_1}{f_2} \right|, \tag{1}$$

where the logarithm in base 2 and the factor 1200 have been used in order to express d in cents [10].

Undoubtedly, the *octave* is the interval which is more generally used and it can be defined as follows:

Definition 1. Given two sounds with frequencies f_1 and f_2, we say that f_2 is an octave higher than f_1 if f_2 is double f_1.

Two notes an octave apart from each other have the same letter-names. This naming corresponds to the fact that notes an octave apart sound like the same note produced at different pitches and not like entirely different notes. Based on this idea, we can define in $\mathbb{R}^+$ (the subset of all the frequencies of all the sounds) a binary equivalence relation, denoted by $\mathcal{R}$, as follows [13]:

$$f_1 \mathcal{R} f_2 \quad \text{if and only if } \exists n \in \mathbb{Z} \quad \text{such that } f_1 = 2^n f_2. \tag{2}$$

Therefore, instead of dealing with $\mathbb{R}^+$, we can analyze the quotient set $\mathbb{R}^+/\mathcal{R}$, which for a given fixed note f_0 (*diapason*) can be identified with the interval $[f_0, 2f_0[$. In 1955 the International Organization for Standardization fixed as diapason or concert pitch the frequency of A_4, the A above middle C, at 440 Hz (see [2]). However, for the sake of simplicity, we will assume that $f_0 = 1$ and work in the interval $[1, 2[$.

Let us introduce the definition of a tuning system

Definition 2. Let f_1/f_2 be an interval and $\lambda = |\log_2(f_1/f_2)|$. We call the tuning system generated by λ the set

$$S_\lambda := \{2^{c_n} \mid c_n = \lambda n - \lfloor \lambda n \rfloor,\ n \in \mathbb{Z}\} \subset [1, 2[, \tag{3}$$

where $\lfloor x \rfloor$ is the integer part of x.

Some systems are generated by more than one interval and in such cases, it is necessary to specify when and how many times each interval appears.

Definition 3. Let $\Lambda = \{\lambda_i\}_{i=1}^k \subset [0, 1[$ and a family of functions $f_i : \mathbb{Z} \to \mathbb{Z}$, $i = 1, 2, \ldots, k$. We call the tuning system generated by the intervals $\{2^{\lambda_i}\}_{i=1}^k$ (or simply by $\{\lambda_i\}_{i=1}^k$) and $\mathcal{F} = \{f_i\}_{i=1}^k$ the set

$$S_\Lambda^{\mathcal{F}} := \left\{ 2^{c_n} \mid c_n = \sum_{i=1}^k \lambda_i f_i(n) - \left\lfloor \sum_{i=1}^k \lambda_i f_i(n) \right\rfloor,\ n \in \mathbb{Z} \right\} \subset [1, 2[. \tag{4}$$

If every element in the tuning system is a rational number, we say that it is a tuned system whereas if some element is an irrational number then the system is a temperament [16].

Remark 1. The advantage of expressing the tuned notes as 2^{c_n} is that if our reference note is 2^0, by (1) the exponent c_n provides the pitch sensation.

Once we fix a tuning system we are able to establish if a sound is tuned or not.

Definition 4. A sound with frequency f is a well-tuned note in $S_\Lambda^{\mathcal{F}}$ if $n \in \mathbb{Z}$ exists such that $2^n \cdot f \in S_\Lambda^{\mathcal{F}}$.

Usually, musicologists feel more comfortable with ordering all or part of the notes in a tuning system by fifths and then representing them as points in a circumference called a *cycle or circle of fifths*. This cycle is not necessarily closed, i.e. the notes are not necessarily repeated, and actually to force the cycle to be closed one or even more fifths must be modified [2,6]. However, in the Equal Temperament, for instance, the circle closes naturally with 12 equal fifths. In order to deal with a circumference instead of an interval $[f_0, 2f_0[$ we consider the function $\varphi : \mathbb{Z} \times S^1 \to S^1$ given by

$$\varphi(n, \theta) = \theta + 2\pi c_n, \quad n \in \mathbb{Z}, \tag{5}$$

where $S^1 = \{\theta \colon \theta \in [0, 2\pi]\}$, and $\{c_n\}_{n \in \mathbb{Z}}$ is the sequence of exponents in (2) and (4), the sequence $\{\varphi(n, 0)\}_{n \in \mathbb{Z}}$ is equivalent to $S_\Lambda^{\mathcal{F}}$.

3. Mathematical translation of some known tuning systems

Among the different tuning systems used in the western music since the 6th century B.C., four of them are specially interesting because they still remain in our classic orchestras [4–6]: the Pythagorean, Zarlinean, Holderean and Equal Tempered systems. In fact, performers consider that they "sound" in the Equal Tempered system. However, some experiments [8] show that these four systems coexist and their simultaneity does not imply any loss of beauty in the ensemble.

In this section, we express these four tuning systems in terms of Definition 4 and we briefly show some of their advantages and disadvantages.

3.1. Tuned systems

(a) *Pythagorean system*: This system is obtained by "transferring" the powers of 3 to the interval $[1,2[$. Each time we multiply (resp. divide) by $\frac{3}{2}$ a frequency f, we say that it goes up (resp. down) by a fifth. It is easy to prove that the Pythagorean system, generated by the fifth interval, or by $\lambda = \log_2\left(\frac{3}{2}\right)$, is the set of notes given by

$$S_\lambda := \{2^{c_n} \mid c_n = n\,\log_2(3/2) - \lfloor n\,\log_2(3/2)\rfloor,\ n \in \mathbb{Z}\}. \tag{6}$$

(b) *Just Intonation*: The Just Intonation can be viewed as a generalization of the Pythagorean system because it not only works with powers of 3, but also with powers of 5. Every time we multiply (resp. divide) a frequency f by $\frac{5}{4}$ it is said that f goes up (resp. down) by one-third.

In practice, the Just Intonation can be obtained by replacing some fifths of the Pythagorean system $\frac{3}{2}$, by syntonic fifths $\frac{40}{27}$ (see [15]). Such a fifth is called syntonic because it differs by a syntonic comma [2] from the Pythagorean fifth, i.e. $\frac{3}{2} : \frac{40}{27} = \frac{81}{80}$. Depending on the number of fifths substituted, a different variant is obtained [6]. In this paper, we use Zarlino's approach which can be described as follows:

$$S^{f_1,f_2}_{\lambda_1,\lambda_2} := \left\{2^{c_n} \mid c_n = \sum_{i=1}^{2} \lambda_i f_i(n) - \left\lfloor \left(\sum_{i=1}^{2} \lambda_i f_i(n)\right)\right\rfloor,\ n \in \mathbb{Z}\right\}, \tag{7}$$

where $\lambda_1 = \log_2\left(\frac{3}{2}\right)$, $\lambda_2 = \log_2\left(\frac{5}{4}\right)$ and the functions

$$f_1(n) = n - 4f_2(n), \quad f_2(n) = \left\lfloor \frac{n+1}{7} \right\rfloor + \left\lfloor \frac{n+4}{7} \right\rfloor.$$

Let us analyze the Pythagorean system and the Just Intonation in the circumference S^1. As the generator intervals are irrational numbers, the sequence of exponents $\{c_n\}_{n\in\mathbb{Z}}$ verifies that for all $\theta \in S^1$, the sequence $\{\theta + 2\pi c_n\}_{n\in\mathbb{Z}}$, is dense in S^1 (see [1]). Therefore, it is easy to show that

Proposition 1. *Let $\{c_n\}_{n\in\mathbb{Z}}$ be the sequence of exponents given in (6) or (7).*

(a) *For $n, m \in \mathbb{Z}$ such that $n \neq m$ then $\theta + 2\pi c_n \neq \theta + 2\pi c_m$.*
(b) *Given $\theta_0, \theta_1 \in S^1$ such that $\theta_1 \notin \{\theta_0 + 2\pi c_n\}_{n\in\mathbb{Z}}$, then*

$$\{\theta_0 + 2\pi c_n\}_{n\in\mathbb{Z}} \cap \{\theta_1 + 2\pi c_n\}_{n\in\mathbb{Z}} = \emptyset.$$

This result shows two disadvantages of the Pythagorean and Zarlinean systems. By (a) the circle of fifth is not closed hence, to establish an appropriate number of notes in an octave, some additional criteria are necessary. According to (b), the point $\theta_0 \in S^1$ determines the sequence $\{\theta_0 + 2\pi c_n\}_{n\in\mathbb{Z}}$, hence given the passage $\{\beta f_1, \beta f_2, \ldots, \beta f_k\}$ obtained by multiplying by β (*transpose down an interval* β) a well-tuned passage $\{f_1, f_2, \ldots, f_k\}$ need not be tuned.

3.2. Tempered systems

The temperaments appear as approximations of the tuned systems in order to avoid the problems described in Proposition 1. As in tempered systems some irrational numbers appear, so some tempered intervals do not correspond to the natural harmonics. However, the many advantages of temperaments have caused the words 'tempered' and 'tuned' to be considered synonymous in current musical practice.

The most used temperaments are the cyclic temperaments that divide the octave into equal parts (in this way, the problems expressed in Proposition 1 are solved). Given a natural number q, the well-tuned notes are

$$T^q := \{2^{k/q}\}_{k=0}^{q-1}. \tag{8}$$

In order to express T^q in terms of Definition 2, it suffices to take into account that given a natural number q, for each $p \in \mathbb{N}^*$ such that $(p,q)=1, p<q$, $T^q = S_{p/q}$ holds.

(c) *Equal temperament* (of 12 notes): This was utilized in at least 1482 by B. Ramos de Pareja in his book (*Música Práctica* [6,15]). However, it was not extended until the appearance of *Das wohltemperierte Klavier I*, (1721) of J.S. Bach. In this temperament the octave is divided into 12 equal parts, $T^{12} = \{2^{k/12}, 0 \leqslant k \leqslant 11\}$, hence we can express T^{12} in terms of Definition 2 as

$$S_{7/12} := \left\{ 2^{c_n} \mid c_n = n\,\frac{7}{12} - \left\lfloor n\,\frac{7}{12} \right\rfloor,\ n \in \mathbb{Z} \right\}. \tag{9}$$

Nowadays, practically all musicians work with this tuning system and, in fact, it is called *The Good Temperament* [6].

(d) *Temperament of Hölder*: Since the 17th century, hundreds of temperaments have arisen, but we will only work with the Hölder's temperament because it is still utilized in many theoretical studies. W. Hölder (1614–1697) proposed a temperament that divides the octave into 53 equal parts, $T^{53} = \{2^{k/53}\}_{k=0}^{52}$. In this way, a very good approximation of the Pythagorean system is obtained. Its notes can be expressed as

$$S_{31/53} := \left\{ 2^{c_n} \mid c_n = n\,\frac{31}{53} - \left\lfloor n\,\frac{31}{53} \right\rfloor,\ n \in \mathbb{Z} \right\}. \tag{10}$$

Notice that the choice of the values $\frac{7}{12}$ and $\frac{31}{53}$ in expressions (9) and (10), respectively, is not unique. Theorem 1 justifies this choice.

In order to illustrate the differences between the tuning systems, let us consider the frequencies of the notes from three measures of the Third Movement of *Music for Strings*, *Percussion and Celesta* (1936) by Béla Bartók (see Fig. 1).

For each note we compute the distance between its frequency in the Equal Tempered system fixing $A_4 = 440$ Hz and the remaining systems (see Fig. 2).

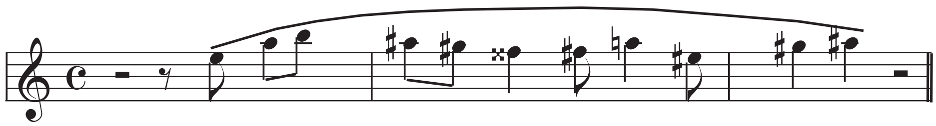

Fig. 1. Fragment of *Music for Strings, Percussion and Celesta* by Béla Bartók.

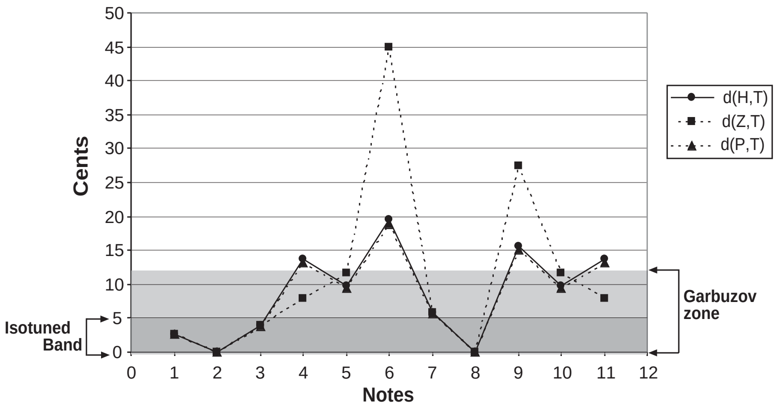

Fig. 2. Distances between the notes in the pentagram for the Holderean, H, Pythagorean, P, and Zarlinean, Z, systems.

Notice that notes whose distance is greater than 5 cents can be distinguished by the human ear ("isotuned band") [15]. Even if we accept as the same note two notes whose distance is less than or equal to 12 cents (Garbuzov zone for the unison [8]), several notes in the fragment analyzed would not be well tuned. As we analyze in Example 3, the distances from the Equal Tempered $F^{\sharp\sharp}$, $E^{\sharp}$ and $A^{\sharp}$ to the same notes for the remaining tuning systems are too big, especially the distances to the Zarlinean system.

4. Some concepts of fuzzy musical notes

A musical note must be understood as a band of frequencies around a "central frequency" f and, as we will show in this section, modelling by means of a fuzzy set $\tilde{f}$ becomes very suitable. The idea of modelling musical notes as fuzzy sets is not new (see [8]) and it can justified for several reasons:

(a) *Technical reasons*: In fast passages, musicians choose comfortable although slightly out of tune positions, while, lip pressure, temperature, humidity, hall acoustics, etc. all modify the frequencies.

(b) *Psychological reasons*: The perception of the intervals is not the same for all of us and even more significantly, it depends on the mood of the performer (see [8,14]).

To be more precise, we will consider a musical sound as a fuzzy number which should reflect the sensation that a frequency f produces, i.e. $\log_2(f)$ (see expression (1)), and whose membership function should model musicians usual practices. With this aim, we will use the information which

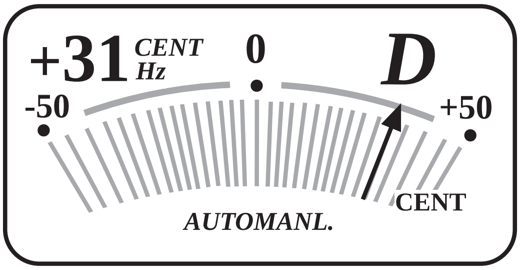

Fig. 3. Scheme of a standard chromatic tuner.

an electronic chromatic tuner provides. These tuners, based on the 12 note tempered system (see Table 2), divide the octave into 12 equal parts. Each part is 100 cents wide, so if we represent it as a segment, the (crisp) tuned note would be in the middle, and the extremes would be obtained by adding and subtracting 50 cents from the central note. As we will see in Example 1, the deviation to the central note gives rise to a membership function.

Example 1. Let us consider an electronic tuner in which we have set $A_4 = 440$ Hz. If it detects a note N whose frequency is 299 Hz, then we would obtain (see Fig. 3):

Note: D, Deviation: $+31.1702$ cents.

Firstly, the tuner locates the tuned note closer to N which, in this case, is D and then it measures the deviation between N and D. This deviation is an indicator of the degree of truthfulness of the statement "N is the note D",

$$1 - \frac{\text{deviation}}{50} = 1 - \frac{31.1702}{50} = 0.3766. \tag{11}$$

Other possibilities for defining this degree of truthfulness could be valid, but the linear choice reflects musicians' usual practices. In general, they consider that 50 cents represents $\frac{1}{4}$ of a tone and, for a note whose deviation is 25 cents, they would say that "this note has deviated by $\frac{1}{8}$ of a tone".

The tuner uses the Equal Temperament, hence in the interval $[1,2[$ the well-tuned notes are $t_n = 2^{n/12}$, $0 \leqslant n \leqslant 11$, which correspond respectively to C, $C^\sharp$, D, $D^\sharp$, E, F, $F^\sharp$, G, $G^\sharp$, A, $A^\sharp$, B. As our interest is in knowing the pitch sensation, we should work with the exponents, i.e. $\log_2(t_n)$, $0 \leqslant n \leqslant 11$ (see Remark 1).

When the diapason is fixed at 440 Hz, note C_4 is determined as $C_4 = 2^{-3/4} \times 440$ Hz, and the reference interval is

$$[f^0, 2f^0[:= [2^{-3/4} \cdot 440, 2^{1/4} \cdot 440[. \tag{12}$$

Once we have established f^0, with the aim of translating each frequency f to the interval $[1,2[$, and subsequently take the exponent corresponding to 2, we need to make use of the following transformation:

$$f^* := \log_2\left(\frac{f}{f^0}\right) - \left\lfloor \log_2\left(\frac{f}{f^0}\right) \right\rfloor. \tag{13}$$

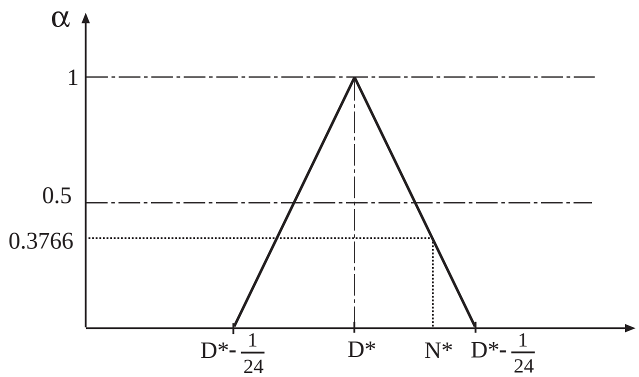

Fig. 4. Membership function of note N in Example 1.

Taking this transformation into account, (11) can be generalized in an operative way considering that note D defines a symmetric triangular fuzzy number $\tilde{D} = \left(D^*, \frac{1}{24}\right) = \left(\frac{1}{6}, \frac{1}{24}\right)$ whose membership function is

$$\mu_{\tilde{D}}(x) = \begin{cases} 1 - 24|D^* - x| & \text{if } |D^* - x| < \frac{1}{24}, \\ 0 & \text{otherwise.} \end{cases} \tag{14}$$

Therefore, the membership degree of N is $\mu_{\tilde{D}}(N) = \mu_{\tilde{D}}(N^*) = 0.3766$ (see Fig. 4). Notice that the membership degree of any note whose distance to D is greater than 50 cents to $\tilde{D}$ is zero.

Following this reasoning we can establish the following definition:

Definition 5. Let $\tilde{t} = (t, \delta)$ be a symmetric triangular fuzzy number, where $t, \delta \in [0, 1]$. The triangular fuzzy number $2^{\tilde{t}} := (2^t, 2^{t-\delta}, 2^{t+\delta})$ whose membership function is

$$\mu_{2^{\tilde{t}}}(x) = \begin{cases} 1 - \dfrac{2^t - x}{2^t - 2^{t-\delta}}, & 2^{t-\delta} < x \leqslant 2^t, \\ 1 - \dfrac{x - 2^t}{2^{t+\delta} - 2^t}, & 2^t < x \leqslant 2^{t+\delta}, \\ 0 & \text{otherwise,} \end{cases} \tag{15}$$

is a fuzzy musical note.

Remark 2. The quantity $\Delta := 1200\delta$ expresses, in cents, the tolerance that we admit. Hence, for a chromatic tuner (based on 12 notes) we have $\delta = \frac{1}{2q} = \frac{1}{24}$, therefore, the tolerance is $\Delta = 1200 \times \frac{1}{24} =$ 50 cents.

The choice of a symmetric triangular membership function is justified by musicians' usual practices as it was in (11).

Once we have stated the concept of a fuzzy musical note, our interest is to determine when two notes sound well together. It is well known that a fuzzy number $\tilde{a}$ can be considered as a possibility distribution and its membership function $\mu_{\tilde{a}}(x)$ can be interpreted as the degree of possibility of the statement "x is in $\tilde{a}$" [3,18]. Therefore the equality of two notes $\tilde{a}$ and $\tilde{b}$ restricted by $\mu_{\tilde{a}}$ and $\mu_{\tilde{b}}$ can be assessed by using the index of consistency

$$\mathrm{Pos}[\tilde{a} = \tilde{b}] := \sup_{x \in E} \min\{\mu_{\tilde{a}}(x), \mu_{\tilde{b}}(x)\} = \sup_{x \in E} \mu_{\tilde{s} \cap \tilde{t}}(x) \tag{16}$$

introduced by Zadeh [17]. Although there is also a degree of intersection between $\mu_{\tilde{a}}$ and $\mu_{\tilde{b}}$, it evaluates to what extent it is possible to find a common value for $\tilde{a}$ and $\tilde{b}$.

Definition 6. Let $2^{\tilde{s}}$ and $2^{\tilde{t}}$ be two musical notes, where $\tilde{s} = (s, \delta)$ and $\tilde{t} = (t, \delta)$. We define the degree of compatibility between $2^{\tilde{s}}$ and $2^{\tilde{t}}$ as

$$\mathrm{Compat}[2^{\tilde{s}}, 2^{\tilde{t}}] := \mathrm{Pos}[\tilde{s} = \tilde{t}] \tag{17}$$

and we say that $2^{\tilde{s}}$ and $2^{\tilde{t}}$ are α-compatible, $\alpha \in [0, 1]$, if $\mathrm{Compat}[2^{\tilde{s}}, 2^{\tilde{t}}] \geqslant \alpha$.

If we say that $2^{\tilde{s}}$ and $2^{\tilde{t}}$ are compatible, we mean to say that they are $\frac{1}{2}$-compatible.

The next proposition allows us to ensure the α-compatibility (see also [5]).

Proposition 2. *Two musical notes* $2^{\tilde{s}}$, $2^{\tilde{t}}$, *where* $\tilde{t} = (t, \delta)$ *and* $\tilde{s} = (s, \delta)$, $\delta > 0$, *are* α*-compatible*, $\alpha \in [0, 1]$, *if and only if* $|t - s| \leqslant 2\delta(1 - \alpha)$.

Proof. We can assume that $s < t$ without any loss of generality. According to (16), when the intersection between $\tilde{s}$ and $\tilde{t}$ is non-empty, $\tilde{s} \cap \tilde{t}$ is the triangular non-normalized fuzzy number whose membership function is

$$\mu_{\tilde{s} \cap \tilde{t}}(x) = \begin{cases} 0 & \text{if } x \leqslant t - \delta, \\ 1 - \dfrac{1}{\delta}(t - x) & \text{if } t - \delta < x \leqslant \dfrac{s + t}{2}, \\ 1 - \dfrac{1}{\delta}(x - s) & \text{if } \dfrac{s + t}{2} < x \leqslant s + \delta, \\ 0 & \text{if } x > s + \delta. \end{cases} \tag{18}$$

Therefore, in general the compatibility between $2^{\tilde{s}}$ and $2^{\tilde{t}}$ is given by

$$\mathrm{Compat}[2^{\tilde{s}}, 2^{\tilde{t}}] = \sup \mu_{\tilde{t} \cap \tilde{s}} = \mu_{\tilde{t} \cap \tilde{s}}\left(\frac{s + t}{2}\right) = \max\left\{0, 1 - \frac{|t - s|}{2\delta}\right\}. \tag{19}$$

Then, $\mathrm{Compat}[2^{\tilde{s}}, 2^{\tilde{t}}] \geqslant \alpha$ if and only if $|t - s| \leqslant 2\delta(1 - \alpha)$. □

In particular, if $\delta = 1/2q$, $q \in \mathbb{N}$, $q \neq 0$ (see Remark 2), for a given $\alpha \in [0, 1]$, $\tilde{s}$ and $\tilde{t}$ are α-compatible iff $|t - s| < (1 - \alpha)/q$, hence they are compatible when

$$|t - s| \leqslant \frac{1}{2q}. \tag{20}$$

Remark 3. It is usually more comfortable to calculate the compatibility between two notes in terms of their frequencies. Hence, given two notes with frequencies f_1 and f_2, for which we admit a tolerance of Δ cents, according to (19) the compatibility between f_1 and f_2 is given by

$$\text{Compat}[\tilde{f}_1, \tilde{f}_2] := \max\left\{0, 1 - \frac{d(f_1, f_2)}{2\Delta}\right\}, \tag{21}$$

where d is their distance expressed in cents (see expression (1)).

Our next purpose is to analyze the compatibility between tuning systems, so we introduce the definition of a fuzzy tuning system.

Definition 7. Let $\delta > 0$, $\Lambda = \{\lambda_i\}_{i=1}^{k} \subset \mathbb{R}^+$ and a family of functions $f_i : \mathbb{Z} \to \mathbb{Z}$, $i = 1, 2, \ldots, k$. We call a fuzzy tuning system generated by the intervals $\{\lambda_i\}_{i=1}^{k}$ and $\mathcal{F} = \{f_i\}_{i=1}^{k}$ to the set

$$\tilde{S}_{\Lambda}^{\mathcal{F}}(\delta) := \left\{2^{\tilde{c}_n} \mid \tilde{c}_n = \left(\sum_{i=1}^{k} \lambda_i f_i(n) - \left\lfloor \sum_{i=1}^{k} \lambda_i f_i(n) \right\rfloor, \delta\right),\ n \in \mathbb{Z}\right\}. \tag{22}$$

Definition 8. Let $\tilde{S}^q(\delta) = \{2^{\tilde{s}_i}\}_{i=1}^{q}$ and $\tilde{T}^q(\delta) = \{2^{\tilde{t}_i}\}_{i=1}^{q}$ be two tuning systems with q notes. We say that $\tilde{S}^q(\delta)$ and $\tilde{T}^q(\delta)$ are α-compatible, $\alpha \in]0, 1]$, if for each $\tilde{s}_i \in \tilde{S}^q(\delta)$ there is a unique $\tilde{t}_j \in \tilde{T}^q(\delta)$ such that

$$\text{Compat}[2^{\tilde{s}_i}, 2^{\tilde{t}_j}] \geqslant \alpha. \tag{23}$$

The quantity α in (23) can be regarded as the degree of interchangeability between $\tilde{S}^q$ and $\tilde{T}^q$. The following result provides us a upper bound of the compatibility level:

Proposition 3. *Let $\tilde{S}^q(\delta) = \{2^{\tilde{s}_i}\}_{i=1}^{q}$ and $\tilde{T}^q(\delta) = \{2^{\tilde{t}_i}\}_{i=1}^{q}$ be two tuning systems α-compatible. Thus, the level of compatibility α verifies*

$$\alpha \leqslant \min_i \left\{1 - \frac{|t_i - s_i|}{2\delta}\right\} = \min_i \{\text{Compat}[2^{\tilde{s}_i}, 2^{\tilde{t}_i}]\}, \tag{24}$$

where for each $\tilde{s}_i = (s_i, \delta)$, the number $\tilde{t}_i = (t_i, \delta)$ is the unique exponent such that $2^{\tilde{s}_i}$ and $2^{\tilde{t}_i}$ are α-compatible.

Proof. By Definition 8, given $2^{\tilde{s}_i} \in \tilde{S}^q(\delta)$ there is a unique $2^{\tilde{t}_i} \in \tilde{T}^q(\delta)$ α-compatible with it. By Proposition 2, $\alpha \leqslant 1 - (|t_i - s_i|)/2\delta$. Thus, this inequality holds for every note in these systems then, $\alpha \leqslant \min_i\{1 - (|t_i - s_i|)/2\delta\}$ and, according to expression (19), $1 - (|t_i - s_i|)/2\delta = \text{Compat}[2^{\tilde{s}_i}, 2^{\tilde{t}_i}]$. □

On the other hand, we can give some sufficient conditions for the non-compatibility as follows:

Proposition 4. *Let $\tilde{S}^q(\delta) = \{2^{\tilde{s}_i}\}_{i=1}^{q}$ and $\tilde{T}^q(\delta) = \{2^{\tilde{t}_i}\}_{i=1}^{q}$ be two tuning systems. Then,*

(a) *If there are $2^{\tilde{s}_i}, 2^{\tilde{s}_k} \in \tilde{S}^q(\delta)$, $\tilde{s}_i \neq \tilde{s}_k$, $2^{\tilde{t}_{k_0}} \in \tilde{T}^q(\delta)$ such that*

$$\text{Compat}[2^{\tilde{t}_{k_0}}, 2^{\tilde{s}_i}] \geqslant \alpha \quad \textit{and} \quad \text{Compat}[2^{\tilde{t}_{k_0}}, 2^{\tilde{s}_k}] \geqslant \alpha,$$

for a given $\alpha \in]0, 1]$, then $\tilde{S}^q(\delta)$ and $\tilde{T}^q(\delta)$ are not α-compatible.

(b) *If there are* $2^{\tilde{s}_i}, 2^{\tilde{s}_k} \in \tilde{S}^q(\delta)$, $\tilde{s}_i \neq \tilde{s}_k$, $2^{\tilde{t}_{k_0}} \in \tilde{T}^q(\delta)$ *verifying*

$$\max_{\tilde{t}_j} \text{Compat}[2^{\tilde{t}_j}, 2^{\tilde{s}_i}] = \text{Compat}[2^{\tilde{t}_{k_0}}, 2^{\tilde{s}_i}] = \beta_1,$$

$$\max_{\tilde{t}_j} \text{Compat}[2^{\tilde{t}_j}, 2^{\tilde{s}_k}] = \text{Compat}[2^{\tilde{t}_{k_0}}, 2^{\tilde{s}_k}] = \beta_2,$$

then $\tilde{S}^q(\delta)$ *and* $\tilde{T}^q(\delta)$ *are not* α*-compatible for any* $\alpha \in]0,1]$.

Proof. (a) Follows from the uniqueness required in Definition 8.

(b) Let us suppose that, for instance, $\beta_1 \leqslant \beta_2$. On one hand, if $\alpha \in]0, \beta_1]$ we have $\text{Compat}[2^{\tilde{t}_{k_0}}, 2^{\tilde{s}_i}] \geqslant \alpha$ and $\text{Compat}[2^{\tilde{t}_{k_0}}, 2^{\tilde{s}_k}] \geqslant \alpha$, therefore, by (a) the systems are not α-compatible. On the other hand, if $\alpha \in]\beta_1, 1]$ as $\beta_1 = \max_{\tilde{t}_j} \text{Compat}[2^{\tilde{t}_j}, 2^{\tilde{s}_i}]$, there is no $2^{\tilde{t}_j} \in \tilde{T}^q(\delta)$ such that $\text{Compat}[2^{\tilde{t}_j}, 2^{\tilde{s}_i}] \geqslant \alpha$. Then, by Definition 8, the systems cannot be α-compatible. □

Notice that the concept of α-compatibility between systems reflects not only the idea of proximity between the notes of two different systems, but also that their configuration is similar. In practice, musicians must know which note is close enough to which other one to be considered as interchangeable and, clearly, this criterion must be unique.

As some tuning systems consist of a finite number of notes, it can happen that two systems were compatible or not depending on which terms are chosen (see (6) and (7)). We will see that in the following example:

Example 2. We consider the Pythagorean system $\tilde{S}(\delta) = \{2^{\tilde{s}_n}\}_{n\in\mathbb{Z}}$ and the Equal Temperament of 41 notes $\tilde{T}^{41}(\delta) = \{2^{\tilde{t}_n}\}_{n=0}^{40}$, where

$$\tilde{s}_n = \left(n \log_2 \frac{3}{2} - \left\lfloor n \log_2 \frac{3}{2} \right\rfloor, \delta\right), \quad \tilde{t}_n = \left(n \frac{24}{41} - \left\lfloor n \frac{24}{41} \right\rfloor, \delta\right).$$

Thus,

(a) $\tilde{S}_1^{41}\left(\frac{1}{2\cdot 41}\right) = \{2^{\tilde{s}_n}\}_{n=-17}^{23}$ and $\tilde{T}^{41}\left(\frac{1}{2\cdot 41}\right) = \{2^{\tilde{t}_n}\}_{n=-17}^{23}$ are α-compatible, for $\alpha \geqslant \frac{1}{2}$.
(b) $\tilde{S}_2^{41}\left(\frac{1}{2\cdot 41}\right) = \{2^{\tilde{s}_n}\}_{n=0}^{40}$ and $\tilde{T}^{41}\left(\frac{1}{2\cdot 41}\right) = \{2^{\tilde{t}_n}\}_{n=0}^{40}$ are not compatible for any $\alpha \in]0,1]$.

By a direct calculus it is easy to prove that if $n \neq m$, $n, m \in \{-17, \ldots, 23\}$, then $|t_n - s_n| \leqslant \frac{1}{82}$, $|t_n - s_m| > \frac{1}{82}$. And so, by applying (20), $\tilde{S}_1^{41}\left(\frac{1}{2\cdot 41}\right)$ and $\tilde{T}^{41}\left(\frac{1}{2\cdot 41}\right)$ are $\frac{1}{2}$-compatible. However, $\tilde{S}_2^{41}\left(\frac{1}{2\cdot 41}\right)$ and $\tilde{T}^{41}\left(\frac{1}{2\cdot 41}\right)$ are not compatible because $2^{\tilde{t}_{40}}$ is the most similar note to both $2^{\tilde{s}_{28}}$ and $2^{\tilde{s}_{40}}$ (see Fig. 5), i.e.

$$\max_{\tilde{t}_j} \text{Compat}[2^{\tilde{t}_j}, 2^{\tilde{s}_{28}}] = \text{Compat}[2^{\tilde{t}_{40}}, 2^{\tilde{s}_{28}}] = 0.661499,$$

$$\max_{\tilde{t}_j} \text{Compat}[2^{\tilde{t}_j}, 2^{\tilde{s}_{40}}] = \text{Compat}[2^{\tilde{t}_{40}}, 2^{\tilde{s}_{40}}] = 0.338502.$$

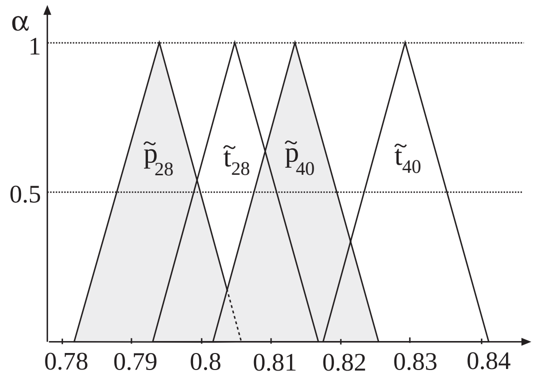

Fig. 5. Membership functions of the notes $2^{\tilde{t}_{28}}, 2^{\tilde{t}_{40}} \in \tilde{T}^{41}\left(\frac{1}{2\cdot 41}\right)$ and $2^{\tilde{s}_{28}}, 2^{\tilde{s}_{40}} \in \tilde{S}_2^{41}\left(\frac{1}{2\cdot 41}\right)$.

5. Sufficient conditions of compatibility

With the aim of obtaining sufficient conditions for the compatibility of tuning systems, let us recall some concepts of continued fractions:

Definition 9. Given $\{a_i\}_{i=0}^{\infty}$, a sequence of natural numbers, where $a_i \neq 0,\ i > 0$, we construct

$$[a_0] = a_0, \quad [a_0, a_1] = a_0 + \frac{1}{a_1}, \quad [a_0, a_1, a_2] = a_0 + \frac{1}{a_1 + 1/a_2} \cdots \tag{25}$$

and we denote $r_n = [a_0, \ldots, a_n] = p_n/q_n$ (if $a_0 = 0$, then $p_0 = 0,\ q_0 = 1$). The sequence $\{r_n\}_{n=0}^{\infty}$ is said to be a *continued fraction* associated to $\{a_i\}_{i=0}^{\infty}$, and each rational number r_n is said to be a *convergent* of the continued fraction.

Each real number λ has a continued fraction $\{r_n\}_{n=0}^{\infty}$ associated to it, and for a given convergent p_n/q_n, if a rational number p/q exists, $(p, q) = 1$, such that $|\lambda - p/q| < |\lambda - p_n/q_n|$, in [1,9], for instance, it is proved that

$$q \geqslant q_n. \tag{26}$$

This property allows us to prove the following lemma:

Lemma 1. *Let p/q be a convergent of the continued fraction of $\lambda \in \mathbb{R}^+$. Thus,*

$$\left\lfloor k\,\frac{p}{q} \right\rfloor = \lfloor k\lambda \rfloor, \quad -q+1 \leqslant k \leqslant q-1. \tag{27}$$

Proof. We consider $\lambda \neq p/q$ (for $\lambda = p/q$ the result is obvious). Let us assume that $\lambda < p/q$. If $k \in \{-q+1, \ldots, q-1\}$ would exist such that $[k\lambda] < [kp/q]$, we would get a contradiction. Let us distinguish two cases:

(a) If $0<k\leqslant q-1$, there exists $m\in\mathbb{N}$ such that $k\lambda<m\leqslant kp/q$, thus $\lambda<m/k\leqslant p/q$. But p/q is a convergent and $k<q$, and so by (26) we obtain a contradiction.
(b) If $-q+1\leqslant k<0$, there exists $m'\in\mathbb{N}$ such that $k\lambda>-m'\geqslant kp/q$ and consequently $\lambda<-m'/k\leqslant p/q$. Taking into account (a) for $-k$, we obtain a contradiction.

And following similar reasoning the case $\lambda>p/q$ can be proved. □

Theorem 1. *Let* $\tilde{S}^q_\lambda=\{2^{\tilde{s}_n}\}_{n=0}^{q-1}$ *be a fuzzy tuning system generated by* λ *and* $\tilde{T}_{p/q}=\{2^{\tilde{t}_n}\}_{n=0}^{q-1}$ *the fuzzy temperament (generated by* p/q*) with* q *notes. If*

$$\left|\lambda-\frac{p}{q}\right|<\frac{1}{2q^2}, \tag{28}$$

then $\tilde{S}^q_\lambda$ *is compatible with* $\tilde{T}_{p/q}$.

Proof. Given $k\in\{0,\dots,q-1\}$, we consider the notes $2^{\tilde{s}_k}$, $2^{\tilde{t}_k}$ whose exponents are, respectively, the symmetric triangular fuzzy numbers

$$\tilde{s}_k=\left(\lambda k-\lfloor\lambda k\rfloor,\frac{1}{2q}\right),\quad \tilde{t}_k=\left(\frac{p}{q}k-\left[\frac{p}{q}k\right],\frac{1}{2q}\right). \tag{29}$$

By Lemma 1 and (28), we obtain

$$|s_k-t_k|=\left|\lambda k-\lfloor\lambda k\rfloor-\frac{p}{q}k+\left\lfloor\frac{p}{q}k\right\rfloor\right|=\left|\lambda k-\frac{p}{q}k\right|=k\left|\lambda-\frac{p}{q}\right|<\frac{1}{2q}.$$

Applying Proposition 2 and (20), $\mathrm{Compat}[2^{\tilde{t}_k},2^{\tilde{s}_k}]>\frac{1}{2}$.

On the other hand, given $k,k'\in\{0,\dots,q-1\}$, $k\neq k'$, let us see that $|s_k-t_{k'}|>1/2q$. If we suppose that $|s_k-t_{k'}|<1/2q$ we would have

$$|t_k-t_{k'}|=|t_k-s_k+s_k-t_{k'}|\leqslant|t_k-s_k|+|s_k-t_{k'}|<\frac{1}{2q}+\frac{1}{2q}=\frac{1}{q},$$

and this is not true because, by construction, for each pair of notes $\tilde{t}_k,\tilde{t}_k{}'\in\tilde{T}_{p/q}$, $|t_k-t_{k'}|\geqslant 1/q$ holds. □

Remark 4. The condition $|\lambda-p/q|<1/2q^2$ given in the above theorem holds easily because at least one of every pair of convergents of the continued fraction of λ verifies this condition. Moreover, for $p,q\in\mathbb{N}$, $(p,q)=1$, verifying (28), p/q is a convergent of the continued fraction of λ (see, for instance [9]).

Actually, Theorem 1 provides us with a constructive method for obtaining cyclic α-compatible temperaments with a given tuning system for $\alpha\geqslant\frac{3}{4}$.

Corollary 1. *Let* $\tilde{S}_\lambda = \{2^{\tilde{s}_n}\}_{n\in\mathbb{Z}}$ *be a tuning system generated by the positive irrational number* λ. *For a given* $p/q \in \mathbb{Q}$, *such that* $|\lambda - p/q| < 1/2q^2$, *the systems* $\tilde{S}^q_\lambda = \{2^{\tilde{s}_n} : \lfloor -q/2 \rfloor + 1 \leqslant n \leqslant \lfloor q/2 \rfloor\}$ *and* $\tilde{T}_{p/q} = \{2^{\tilde{s}_n} : \lfloor -q/2 \rfloor + 1 \leqslant n \leqslant \lfloor q/2 \rfloor\}$, *where* $\tilde{s}_n$ *and* $\tilde{t}_n$ *are given in* (29), *are* $\frac{3}{4}$*-compatible.*

Proof. For a $k \in \{[-q/2]+1, \ldots, [q/2]\}$, as $|\lambda - p/q| < 1/2q^2$, by Lemma 1 we obtain

$$|s_k - t_k| = k\left|\lambda - \frac{p}{q}\right| < \frac{q}{2}\frac{1}{2q^2} = \frac{1}{4q},$$

and with the same arguments as Theorem 1 we obtain the result. □

However, these reasonings are not valid when the tuning system is generated by more than one interval. In this case, finding a cyclic temperament associated to the tuning system means appropriate divisions of the octave such that all the intervals can be approximated. In [2] a possible solution to this question is proposed:

Theorem 2. *If* $\lambda_1, \lambda_2, \ldots, \lambda_k$ *are real numbers, and at least one of them is irrational, then there exist an infinite number of ways of choosing a denominator* q *and numerators* $p_1, p_2, \ldots, q_k$ *in such a way that the approximations*

$$\frac{p_1}{q} \approx \lambda_1,\ \frac{p_2}{q} \approx \lambda_2, \ldots, \frac{p_k}{q} \approx \lambda_k,$$

have the property that the errors are all less than $1/q^{1+1/k}$.

Therefore, the theoretical issue could be solved by using $\{2^{n/q}\}_{n=0}^{q-1}$. However, the denominator q in Theorem 2 is not obtained by means of any constructive method. Hence, it is necessary to make use of other strategies which usually provide good approximations. Let us see what happens with the Just Intonation.

In Section 3.1, we have seen that this tuning system is generated by

$$\lambda_1 = \log_2\left(\frac{3}{2}\right), \quad \lambda_2 = \log_2\left(\frac{5}{4}\right)$$

and also that it can be obtained by modifying some of the terms of the Pythagorean system. Actually, 5 of each 7 terms are obtained by $\frac{3}{2}$ and the other ones with $\frac{40}{27}$. The idea is to calculate an intermediate interval in which the quantities $\frac{3}{2}$ and $\frac{40}{27}$ appear in the proportions above mentioned, i.e.

$$\sqrt[7]{\left(\frac{40}{27}\right)^2\left(\frac{3}{2}\right)^5} = \sqrt[7]{\frac{50}{3}}. \tag{30}$$

In this way a meantone temperament arises, $\tilde{S}_\lambda$, with $\lambda = \log_2 \sqrt[7]{\frac{50}{3}}$, called temperament of $\frac{2}{7}$ of comma [6]. We can apply Corollary 1 for the system $\tilde{S}_\lambda$, i.e. we calculate some convergents of the continued fraction of λ,

$$0 < \frac{1}{2} < \frac{4}{7} < \frac{11}{19} < \frac{69}{119} < \cdots < \lambda < \cdots < \frac{443}{764} < \frac{29}{50} < \frac{7}{12} < \frac{3}{5} < 1, \tag{31}$$

and the denominator values provide us with some possibilities for the cyclic temperament $\tilde{T}$ that approximates to $\tilde{S}_\lambda$, and also analyze the α-compatibility between $\tilde{T}$ and the Zarlinean system.

The fraction $\frac{7}{12}$ is a convergent of the continued fraction of the interval which generates the Pytagorean, Holderean and Equal Temperated systems. In contrast, for the Zarlinean system, the fraction $\frac{7}{12}$ is a convergent of $\lambda = \frac{1}{7}\log_2\left(\frac{50}{3}\right)$ (which is able to approximate to the Zarlinean system). This circumstance allows us to analyze the "goodness" of the sufficient conditions for α-compatibility in the case of 12 notes.

By using Proposition 3 and Table 3 in appendix, we see that for the Pythagorean, Holderean and Equal Temperated systems, the level of compatibility reaches the value 0.8827. However, the level of compatibility for the Zarlinean system is lower: 0.5699, 0.5733 and 0.6677 for the systems of Pythagoras, Hölder and Equal Temperament, respectively.

Finally, in Example 3 we analyze the α-compatibility of the fuzzy notes in the pentagram in Fig. 1. We assume a tolerance of $\Delta = 25$ cents for all of them (it is approximately the double of the Garbuzov zone for the union [8]).

Example 3. Let us consider that the notes in Fig. 1 are in the Equal Temperament $\tilde{T}^{12}(\delta)$ with $\delta = \frac{\Delta}{1200} = \frac{1}{48}$.

Table 1 shows the term of each tuning system corresponding to each note (column 2), i.e.

$$d_i = 2 * C * c_i, \quad i \in \mathbb{Z},$$

where C is the frequency of C in each system (column 2), and c_i is the ith term of the sequence that generates the tuning system (see expressions (6), (7), (9) and (10)). For each tuning system, the distances between the peaks of the fuzzy notes and the Equal Temperament appear in columns 3, 5 and 7. And, according to (21), we calculate the compatibility between the notes as $\mathrm{Compat}[\tilde{f}_1, \tilde{f}_2] = \max\{0, 1 - d(f_1, f_2)/2\Delta\}$ (columns 4, 6 and 8).

Table 1
Distances and compatibilities between the Equal Temperament and the Pythagorean, Zarlinean and Holderean systems for the notes in the pentagram described in Fig. 1

Note	Term	Pythagorean		Zarlinean		Holderean	
		$d(p,t)$	Compat$[p,t]$	$d(z,t)$	Compat$[z,t]$	$d(h,t)$	Compat$[h,t]$
E	d_4	2.6251	0.8950	2.6251	0.8950	2.5464	0.8981
A	d_3	0	1	0	1	0	1
B	d_5	3.9216	0.8431	3.9216	0.8431	3.7817	0.8487
$A^\sharp$	d_{11}	13.6868	0.4525	7.8184	0.6873	13.2151	0.4714
$G^\sharp$	d_8	9.7769	0.6089	11.7284	0.5309	9.4411	0.6224
$F^{\sharp\sharp}$	d_{13}	19.5491	0.2180	44.9689	<u>0</u>	18.8634	0.2455
$F^\sharp$	d_6	5.8646	0.7655	5.8646	0.7654	5.6548	0.7738
A	d_3	0	1	0	1	0	1
$E^\sharp$	d_{11}	15.6413	0.3743	27.3714	<u>0</u>	15.0886	0.3965
$G^\sharp$	d_8	9.7769	0.6089	11.7284	0.5309	9.4411	0.6224
$A^\sharp$	d_{10}	13.6868	0.4525	7.8185	0.6873	13.2151	0.4714

By Proposition 3 the compatibility between two notes is an upper bound for their α-compatibility, i.e. two notes with frequencies f_1 and f_2 are α-compatible for $\alpha \leqslant \mathrm{Compat}[\tilde{f}_1, \tilde{f}_2]$ (see (21)). Thus, for instance, the Zarlinean and Equal Tempered $F^{\sharp\sharp}$ are not α-compatible for any $\alpha \in]0, 1]$. Actually they do not sound together well.

6. Conclusions

Most of the musicians who constitute a classic orchestra must adjust their instrument to obtain a good tuning. For example, wind instruments players modify the air pressure or the finger positions to adapt their notes to the ensemble. Because of this, many musicians feel that the mathematical arguments that justify the tuning systems are impractical.

With the same arguments employed when a chromatic tuner is used, we make the concept of musical note flexible. In this framework, fuzzy mathematical rules and practice are the same thing. In fact, the adjustments that the musicians make, constitute a method for increasing the compatibility level among systems. In this way, describing the tuning systems as fuzzy sets permits us to include in a mathematical structure the daily reality of musicians and their theoretical instruction. In my opinion, this constitutes a good model of reality.

From the idea of α-compatibility, the possibility of substituting a tuning system with another one arises. Therefore, when a tuning system presents many harmonic difficulties such as not allowing certain transpositions, we can use a compatible system to avoid these disadvantages. On the other hand, knowing the compatibility between notes allows musicians to improve their performances by choosing between different tune positions, increasing lip pressures, etc. In fact, our current research is devoted to designing a user-friendly computer program which calculates the compatibility using records as its input.

Finally, we would like to remark that our methods to ensure the α-compatibility are constructive. Moreover, they allow us to determine an appropriate number of divisions of the octave for every tuning system.

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Appendix

In Table 2, we show the frequencies in Hertz for the more usual (crisp) notes in the octave C_4, fixing $A_4 = 440$ Hz. Notice that in the Zarlinean system the order of sharps and flats is opposite to that of the other systems.

As the four systems have been generated by intervals of fifths, it is practical to group them into groups of size seven and order them as $F - C - G - D - A - E - B$. Hence, from c_{-1} to c_5 the natural notes appear, from c_6 to c_{12} the notes with one sharp, from c_{13} to c_{19} the two sharp notes,

Table 2
Frequencies in Hertz for the more usual notes in four tuning systems

Note	Pythagorean system	Equal Temperament	Hölderean system	Note	Zarlinean system
C	260.7407	261.6256	260.7716	C	264
$D^\flat$	274.6898		274.7764	$C^\sharp$	275
$C^\sharp$	278.4375	277.1826	278.3936	$D^\flat$	285.1200
D	293.3333	293.6648	293.3449	D	297
$E^\flat$	309.0261		309.0991	$D^\sharp$	309.3750
$D^\sharp$	313.2422	311.1270	313.1681	$E^\flat$	316.8000
E	330	329.6275	329.9870	E	330
F	347.6543	349.2282	347.7091	F	352
$G^\flat$	366.2531		366.3830	$F^\sharp$	366.6667
$F^\sharp$	371.2500	369.9944	371.2061	$G^\flat$	380.1600
G	391.1111	391.9954	391.1419	G	396
$A^\flat$	412.0347		412.1484	$G^\sharp$	412.5000
$G^\sharp$	417.6562	415.3047	417.5739	$A^\flat$	422.4000
A	440	440	440	A	440
$B^\flat$	463.5391		463.6304	$A^\sharp$	458.3000
$A^\sharp$	469.8633	466.1638	469.7337	$B^\flat$	475.2000
B	495	493.8833	494.9610	B	495

Table 3
Distances in cents between 12 notes

Notes	$d(\mathrm{P},\mathrm{T})$	$d(\mathrm{Z},\mathrm{T})$	$d(\mathrm{H},\mathrm{T})$	$d(\mathrm{H},\mathrm{P})$	$d(\mathrm{Z},\mathrm{P})$	$d(\mathrm{H},\mathrm{Z})$
C	5.86	15.64	5.66	0.20	21.51	21.30
$C^\sharp$	7.82	13.69	7.55	0.27	21.51	21.23
D	1.95	1.95	1.89	0.7	0	0.07
$E^\flat$	11.73	$\underline{31.28}$	11.32	0.41	$\underline{43.01}$	$\underline{42.60}$
E	1.95	1.95	1.89	0.07	0	0.07
F	7.82	13.69	7.55	0.27	21.51	21.23
$F^\sharp$	5.86	15.64	5.66	0.20	21.51	21.30
G	3.91	17.60	3.77	0.14	21.51	21.37
$G^\sharp$	9.77	11.73	9.43	0.34	21.51	21.16
A	0	0	0	0	0	0
$B^\flat$	9.77	$\underline{33.23}$	9.44	0.34	$\underline{43.01}$	$\underline{42.67}$
B	3.91	3.91	3.77	0.14	0	0.14

Notice that we have underlined the distances greater than 25 cents.

etc. If we consider the negative subscripts, from c_{-8} to c_{-2} we obtain the notes with one flat, from c_{-15} to c_{-9} the two flat notes, and so on.

In western music it is usual to employ 12 notes, C, $C^\sharp$, D, $E^\flat$, E, F, $F^\sharp$, G, $G^\sharp$, A, $B^\flat$, and B, which correspond to the terms for $n=-2$ to $n=9$ of Pythagorean (P), Zarlinean (Z), Holderean (H) and Equal Temperament (T) tuning systems. If in the four systems we fix $A_4=440$ Hz, the distances between the (crisp) notes of these systems, in cents, are in Table 3.

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